MATH 2020 Advanced Calculus II

Tutorial 4

Oct 3

Find the volume of the solid S where S is bounded by

1. $z = 0, z = 3 - x, x = 0, y = 1, y = x$;
2. $z = y, r = \cos\theta, y = 0$;
3. $\rho = 1, z = \frac{1}{2}$ (the upper part);
4. $z = x^2 + y^2, z = 0, x^2 + y^2 = 4$;
5. $\rho = 1, z = x^2 + y^2$.

Solutions.

1.

$$\begin{aligned}
\text{volume} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{\frac{1}{\sin\theta}} \int_{0}^{3-r\cos\theta} dz(rdrd\theta) \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{\frac{1}{\sin\theta}} (3 - r\cos\theta) rdrd\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{3}{2}\csc^2\theta - \frac{1}{3}\csc^3\theta\cos\theta\right] d\theta \\
&= \left[\frac{3}{2}(-\cot\theta) + \frac{1}{3}\left(\frac{1}{2}\cot^2\theta\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{4}{3}
\end{aligned}$$

2.

$$\begin{aligned}
\text{volume} &= \int_{0}^{\frac{\pi}{2}} \int_{0}^{\cos\theta} r\sin\theta \cdot rdrd\theta \\
&= \int_{0}^{\frac{\pi}{2}} \frac{1}{3}\cos^3\theta\sin\theta d\theta \\
&= \left[-\frac{1}{12}\cos^4\theta\right]_{0}^{\frac{\pi}{2}} \\
&= \frac{1}{12}
\end{aligned}$$

3.

$$\begin{aligned}\text{volume} &= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\frac{1}{2\cos\phi}}^{1} \rho^2 \sin\phi d\rho d\phi d\theta \\ &= 2\pi \int_0^{\frac{\pi}{3}} \frac{1}{3}\left(1 - \frac{1}{8\cos^3\phi}\right)\sin\phi d\phi \\ &= \frac{2\pi}{3}\left[-\cos\phi - \frac{1}{16\cos^2\phi}\right]_0^{\frac{\pi}{3}} \\ &= \frac{5\pi}{24}\end{aligned}$$

4.

$$\begin{aligned}\text{volume} &= \int_0^{2\pi}\int_0^{2}\int_0^{r^2} dz(rdrd\theta) \\ &= 2\pi\int_0^2 r^3 dr \\ &= 8\pi\end{aligned}$$

5. Notice that the surfaces $\rho = \sqrt{z^2 + r^2} = 1$ and $z = r^2$ intersect along a circle with center on the z-axis and of radius r_0 where r_0 is the unique positive solution to $r^4 + r^2 = 1$, i.e $r_0 = \sqrt{\frac{\sqrt{5}-1}{2}}$.

$$\begin{aligned}\text{volume} &= \int_0^{2\pi}\int_0^{r_0}\left[\sqrt{1-r^2} - r^2\right](rdrd\theta) \\ &= 2\pi\left[-\frac{\sqrt{1-r^2}^3}{3} - \frac{r^4}{4}\right]_0^{r_0} \\ &= 2\pi\left(\frac{1}{3} - \frac{r_0^6}{3} - \frac{r_0^4}{4}\right) \\ &= \frac{5\pi}{12}(3-\sqrt{5})\end{aligned}$$